when $d = 0$, $$A = \begin{pmatrix} \alpha_{1,0}(1) & \cdots & \alpha_{n,0}(1) \\ \vdots & & \\ \alpha_{1,0}(m) & \cdots & \alpha_{n,0}(m) \end{pmatrix}$$

$$\alpha_{i,0}(j) = 1 \quad \text{if} \quad t_j \in [\tau_i, \tau_{i+1})$$

~~for $\mu > 0$, $[\tau_\mu, \tau_{\mu+1})$,~~ In other words, for row j, there is only one i such that $t_j \in [\tau_i, \tau_{i+1})$, and the other entries are all zero.

when $d > 0$, $$A = \begin{pmatrix} \alpha_{1,d}(1) & \alpha_{2,d}(1) & \cdots & \alpha_{n,d}(1) \\ \alpha_{1,d}(2) & \alpha_{2,d}(2) & \cdots & \alpha_{n,d}(2) \\ \vdots & \vdots & & \vdots \\ \alpha_{1,d}(m) & \cdots & \cdots & \alpha_{n,d}(m) \end{pmatrix}$$

Similarly, for row j, if μ is such that $t_j \in [\tau_\mu, \tau_{\mu+1})$, then $\alpha_{i,d}(j)$ is zero if i does not belong in $\alpha_{\mu-d,d}(j)$, $\alpha_{\mu-d+1,d}(j)$, $\cdots$, $\alpha_{\mu,d}(j)$.

Furthermore, $$\left(\alpha_{\mu-d,d}(j)\ \alpha_{\mu-d+1,d}(j) \cdots \alpha_{\mu,d}(j)\right)$$
$$= R^{\mu}_{1,\tau}(t_{j+1})\, R^{\mu}_{2,\tau}(t_{j+2}) \cdots R^{\mu}_{d,\tau}(t_{j+d})$$
$$= \mathbb{R}^{\mu}_{d,\tau}(t)$$

the reason that $\alpha_{i,d}(j)$ is zero if $i \notin (\mu-d, \cdots, \mu)$ because

No.

Date 16 · 8 · 2022

B-spline

$$B_{i,d,\tau}(x) = 0 \quad , \quad x \in [t_j, t_{j+1}) \quad , \quad i \notin (\mu-d, \cdots, \mu)$$

$$\therefore \; 0 = \sum_{l=j-d}^{j} \alpha_{i,d}(l) B_{l,d,t}(x)$$

because $B_{l,d,t}(x)$ are independent,

$$\therefore \; \alpha_{i,d}(l) = 0 \quad \forall \, l \in (j-d, \cdots, j)$$

in particular, for row j, $\alpha_{i,d}(j)$ is zero if $i \notin (\mu-d, \cdots, \mu)$

---

let's get back to the previous example.

let $\mu = 3$, $\tau_\mu = -1$, $\tau_{\mu+1} = 0$

$$\begin{pmatrix} \alpha_{\mu-2,2}(j) & \alpha_{\mu-1,2}(j) & \alpha_{\mu,2}(j) \end{pmatrix}$$

$$= R^{\mu}_{1,\tau}(t_{j+1}) \, R^{\mu}_{2,\tau}(t_{j+2})$$

$$= \begin{pmatrix} \frac{\tau_{\mu+1} - t_{j+1}}{\tau_{\mu+1} - \tau_\mu} & \frac{t_{j+1} - \tau_\mu}{\tau_{\mu+1} - \tau_\mu} \end{pmatrix} \begin{pmatrix} \frac{\tau_{\mu+1} - t_{j+2}}{\tau_{\mu+1} - \tau_{\mu-1}} & \frac{t_{j+2} - \tau_{\mu-1}}{\tau_{\mu+1} - \tau_{\mu-1}} & 0 \\ 0 & \frac{\tau_{\mu+2} - t_{j+2}}{\tau_{\mu+2} - \tau_\mu} & \frac{t_{j+2} - \tau_\mu}{\tau_{\mu+2} - \tau_\mu} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-t_{j+1}}{0-(-1)} & \frac{t_{j+1}+1}{0-(-1)} \end{pmatrix} \begin{pmatrix} -t_{j+2} & t_{j+2}+1 & 0 \\ 0 & \frac{1-t_{j+2}}{2} & \frac{t_{j+2}+1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -t_{j+1} & t_{j+1}+1 \end{pmatrix} \begin{pmatrix} -t_{j+2} & t_{j+2}+1 & 0 \\ 0 & \frac{1-t_{j+2}}{2} & \frac{1+t_{j+2}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} t_{j+1}t_{j+2} & \frac{1 - t_{j+1} - t_{j+2} - 3t_{j+1}t_{j+2}}{2} & \frac{(1+t_{j+1})(1+t_{j+2})}{2} \end{pmatrix}$$

No.

Date 16·8·2022 B-Spline

$j=1$, $(\alpha_{\mu-2,2}(1) \quad \alpha_{\mu-1,2}(1) \quad \alpha_{\mu,2}(1))$ $\quad t_2 = t_3 = -1$

$= (1 \quad 0 \quad 0)$

$j=2: \left(\frac{1}{2} \quad \frac{1}{2} \quad 0\right)$ $\quad t_3 = -1, t_4 = -\frac{1}{2}$

$j=3: \left(0 \quad \frac{3}{4} \quad \frac{1}{4}\right)$ $\quad t_4 = -\frac{1}{2}, t_5 = 0$

similarly for $\mu = 4$, $\tau_\mu = 0$, $\tau_{\mu+1} = 1$

$$(\alpha_{\mu-2,2}(j) \quad \alpha_{\mu-1,2}(j) \quad \alpha_{\mu,2}(j))$$

$$= R^\mu_{1,2}(t_{j+1}) R^\mu_{2,2}(t_{j+2})$$

$$= (1-t_{j+1} \quad t_{j+1}) \begin{pmatrix} \frac{1-t_{j+2}}{2} & \frac{1+t_{j+2}}{2} & 0 \\ 0 & 1-t_{j+2} & t_{j+2} \end{pmatrix}$$

$$= \left(\tfrac{1}{2}(1-t_{j+1})(1-t_{j+2}) \quad \frac{1+t_{j+1}+t_{j+2}-3t_{j+1}t_{j+2}}{2} \quad t_{j+1}t_{j+2}\right)$$

$\therefore$ $j=4: \left(\frac{1}{4} \quad \frac{3}{4} \quad 0\right)$ $\quad t_5 = 0, t_6 = \frac{1}{2}$

$j=5: \left(0 \quad \frac{1}{2} \quad \frac{1}{2}\right)$ $\quad t_6 = \frac{1}{2}, t_7 = 1$

$j=6: (0 \quad 0 \quad 1)$ $\quad t_7 = 1, t_8 = 1$

we can also do it at $j=3$,

$j=3: \left(\frac{3}{4} \quad \frac{1}{4} \quad 0\right)$ $\quad t_4 = -\frac{1}{2}, t_5 = 0$

$$\therefore A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 1/4 & 3/4 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

No.

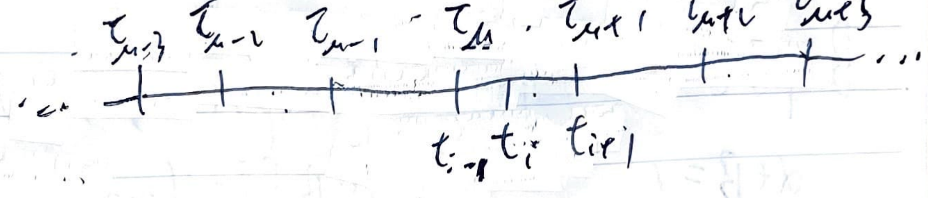
Date 16 · 8 · 2022

B-spline

note that from previous example, for each row, there are $r+1$ non-zero if $t_{j+1}, t_{j+2},$ there are r new knots.

recall that $b = Ac$

$b_j = \sum \alpha_i(j) c_i$



$$f_\mu = \sum_{i=\mu-d}^{\mu} c_i B_{i,d,\tau} = R^\mu_{1,\tau}(x) R^\mu_{2,\tau}(x) \cdots R^\mu_{d,\tau}(x) C^\mu_d$$

where $C^\mu_d = (c_{\mu-d}, \cdots, c_\mu)^T$

from the conversion formula between two B-spline space $(\tau \to t)$,

$b_i = R^\mu_{1,\tau}(t_{i+1}) \cdots R^\mu_{d,\tau}(t_{i+d}) C^\mu_d$

if $t_i \in [\tau_\mu, \tau_{\mu+1})$
$(\alpha_{\mu-d,d}(i) \ldots \alpha_{\mu,d}(i))$
$= R^\mu_{1,\tau}(t_{i+1}) \cdots R^\mu_{d,\tau}(t_{i+d})$
are possibly non-zero

if $t = \tau$,

$c_i = R^\mu_{1,\tau}(\tau_{i+1}) \cdots R^\mu_{d,\tau}(\tau_{i+d}) C^\mu_d$

Blossom

$f(x) = a + bx + cx^2$

$B[f](x_1, x_2) = a + b(x_1 + x_2) + c x_1 x_2$

$g(x) = a + bx + cx^2 + dx^3$

$B[g](x_1, x_2, x_3) = a + b(x_1 + x_2 + x_3) + c(x_1 x_2 + x_2 x_3 + x_1 x_3)$
$+ d x_1 x_2 x_3$

these are called blossoming of polynomial

No.

Date 17 · 8 · 2022

B-spline

Blossom definition:

$$B[p](x_1, \cdots, x_d) = B[p](x_{\pi_1}, \cdots, x_{\pi_d})$$ [Symmetry]

π are permutation group

$$B[p](\cdots, \alpha x + \beta y, \cdots) = \alpha B[p](\cdots, x, \cdots) + \beta B[p](\cdots, y, \cdots)$$ [Affine]

$\alpha + \beta = 1$

$$B[p](x, \cdots, x) = p(x)$$ [Diagonal]

Theorem: Every polynomial p has a unique blossoming.

proof: let's begin with an affine function

$$F(x_1, \cdots, x_d) = C_0 + \sum_{i=1}^{d} \sum_{1 \le j_1 \le \cdots \le j_i \le d} C_{j_1, \cdots, j_i} x_{j_1} x_{j_2} \cdots x_{j_i}$$

for $d = 1$, $F(x_1) = C_0 + C_1 x_1$

$$F(\alpha x + \beta y) = C_0 + C_1(\alpha x + \beta y)$$
$$= (\alpha + \beta) C_0 + \alpha C_1 x + \beta C_1 y$$
$$= \alpha F(x) + \beta F(y)$$

$F(x)$ is symmetry and diagonal,

$\therefore$ blossoming is unique and exists for $d = 1$

for general d, note that if let it have symmetry property

~~$F(x_1, \cdots, x_d)$~~

$$F(1, 0, \cdots, 0) = F(0, 1, \cdots, 0) = F(0, \cdots, 0, 1)$$

$$C_0 + C_1 = C_0 + C_2 = \cdots = C_0 + C_d, \therefore C_1 = C_2 = \cdots = C_d$$

B-spline

No.

Date 17 · 8 · 202

by induction,

$$F(0,\underbrace{1,1,\cdots}_{k},0) = \cdots = F(0,0,\cdots,\underbrace{1,1}_{k})$$

$$p_{k-1} + c_{1,\dots,k} = \cdots = p_{k-1} + c_{d-k+1,\dots,d}, \therefore \; c_{1,\dots,k} = \cdots = c_{d-k+1,\dots,d}$$

$$\therefore F(x_1,\cdots,x_d) = c_0 + \sum_{i=1}^{d} c_i \sum_{j_1,\cdots,j_i} x_{j_1}\cdots x_{j_i}$$

affine and diagonal properties follow easily. //

it is easy to see that

$$B[a_1p_1 + a_2p_2](x_1,\cdots,x_d)$$
$$= a_1 B[p_1](x_1,\cdots,x_d) + a_2 B[p_2](x_1,\cdots,x_d)$$

recall that $R^{\mu}_{d_1,\tau}(x) R^{\mu}_{d_2,\tau}(y) = R^{\mu}_{d_1,\tau}(y) R^{\mu}_{d_2,\tau}(x)$

$\therefore$ the function $G(x_1,\cdots,x_d) = R^{\mu}_{1,\tau}(x_1)\cdots R^{\mu}_{d,\tau}(x_d) C^{\mu}_{d}$ is symmetry.

Furthermore, $G(x_1,\cdots,\alpha x+\beta y,\cdots,x_d) = \alpha G(x_1,\cdots,x,\cdots x_d)$
$+ \beta G(x_1,\cdots,y,\cdots,x_d)$

$$G(x,\cdots,x) = f_{\mu}(x) = \sum_{i=\mu-d}^{\mu} c_i B_{i,d,\tau}(x)$$

$$\therefore \begin{pmatrix} a+b_1x & c_1+d_1x \\ & a_2+b_2x \; \cdots \end{pmatrix} = R^{\mu}_{i,\tau}(X_i) \text{ is linear}$$

No.

Date 17 · 8 · 2022

B-spline

$\therefore$ $G(x_1, \cdots, x_d) = B[f_\mu](x_1, \cdots, x_d) = R^\mu_{1,\tau}(x_1) \cdots R^\mu_{d,\tau}(x_d) C^\mu_d$

is a blossoming.

$\therefore$ $C_j = B[f_\mu](\tau_{j+1}, \cdots, \tau_{j+d})$

if we fix j, and let $\mu = (j, j+1, \cdots, j+d)$

$f_k = \sum_{i=k-d}^{k} c_i B_{i,d,\tau}$, $c_j B_{j,d,\tau}$ is one of RHS,

by continuity of B-spline,

$\therefore$ $C_j = B[f_k](\tau_{j+1}, \cdots, \tau_{j+d})$ for $k = j, j+1, \cdots, j+d$

Lemma : $B_x[(y-x)^k](x_1, \cdots, x_d) = \frac{k!}{d!} D^{d-k}(y-x_1)\cdots(y-x_d)$

$B_x[(y_1-x)\cdots(y_\ell-x)](x_1, \cdots, x_d) = \frac{(d-\ell)!}{d!} \sum_{1\le i_j \le d} (y_1-x_{i_1})\cdots(y_\ell-x_{i_\ell})$

proof : when $k=d$,

$B_x[(y-x)^d](x_1, \cdots, x_d) = (y-x_1)\cdots(y-x_d)$

$B_x[(y-x)^{d-1}](x_1, \cdots, x_d) = \frac{1}{d} D(y-x_1)\cdots(y-x_d)$

$B_x[(y-x)^{d-2}](x_1, \cdots, x_d) = \frac{1}{d(d-1)} D^2(y-x_1)\cdots(y-x_d)$

$\vdots$

$B_x[(y_1-x)](x_1, \cdots, x_d) = \frac{1}{d} \sum_{i=1}^{d}(y_1-x_i)$

$B[(y_1-x)(y_2-x)](x_1, \cdots, x_d) = \frac{1}{d(d-1)} \sum_{1\le i,j\le d}^{i\neq j} (y_1-x_i)(y_2-x_j)$

$\vdots$

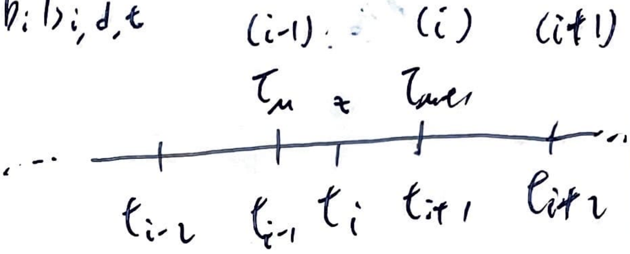
//

B-spline

17-8-2022

Böhm algorithm: $f = \sum_{i=1}^{n} c_i B_{i,d,\tau} = \sum_{i=1}^{n+1} b_i B_{i,d,t}$

insertion of a knot at $[\tau_\mu, \tau_{\mu+1})$



$\mu = i-1$

for $j \leq \mu - d$, $j \leq k \leq j+d$

$$b_j = B[f_k](t_{j+1}, \cdots, t_{j+d}) = B[f_k](\tau_{j+1}, \cdots, \tau_{j+d}) = c_j$$

similarly for $j \geq i$, $l \geq \mu+1$

Ex, $j = i$, $l = \mu+1$, $t_{j+1} = \tau_{\mu+1}$,

$$b_j = B[f_k](t_{j+1}, \cdots, t_{j+d}) = B[f_k](\tau_{l}, \cdots, \tau_{l+d-1}) = c_l = c_{j-1}$$

for $\mu - d + 1 \leq j \leq \mu$, t_i or z appears in $(t_{j+1}, \cdots, t_{j+d})$

$$\therefore b_j = B[f_k](t_{j+1}, \cdots, z, \cdots, t_{j+d})$$

let $k = \mu$,

$$b_j = B[f_\mu](t_{j+1}, \cdots, z, \cdots, t_{j+d}) = B[f_\mu](\tau_{j+1}, \cdots, z, \cdots, \tau_{j+d-1})$$

Ex. $j = \mu - d + 1$, $B[f_\mu](t_{j+1}, \cdots, t_{i-1}, t_i \text{ or } z)$, $B[f_\mu](\tau_{j+1}, \cdots, \tau_\mu, t_i \text{ or } z)$

$j = \mu - d + 2$, $B[f_\mu](t_{j+1}, \cdots, t_{i-1}, z, t_{i+1})$, $B[f_\mu](\tau_{j+1}, \cdots, \tau_\mu, z, \tau_{\mu+1})$

$\underline{j+d-1 = \mu+1}$

write $z = \frac{\tau_{j+d} - z}{\tau_{j+d} - \tau_j} \tau_j + \frac{z - \tau_j}{\tau_{j+d} - \tau_j} \tau_{j+d}$

$$\therefore b_j = B[f_\mu]\left(\tau_{j+1}, \cdots, \frac{\tau_{j+d} - z}{\tau_{j+d} - \tau_j} \tau_j + \frac{z - \tau_j}{\tau_{j+d} - \tau_j} \tau_{j+d}, \cdots, \tau_{j+d-1}\right)$$

$$= \frac{\tau_{j+d} - z}{\tau_{j+d} - \tau_j} B[f_\mu](\tau_{j+1}, \cdots, \tau_j, \cdots, \tau_{j+d-1})$$

$$+ \frac{z - \tau_j}{\tau_{j+d} - \tau_j} B[f_\mu](\tau_{j+1}, \cdots, \tau_{j+d})$$

$$= \frac{z - \tau_j}{\tau_{j+d} - \tau_j} c_j + \frac{\tau_{j+d} - z}{\tau_{j+d} - \tau_j} c_{j-1}$$

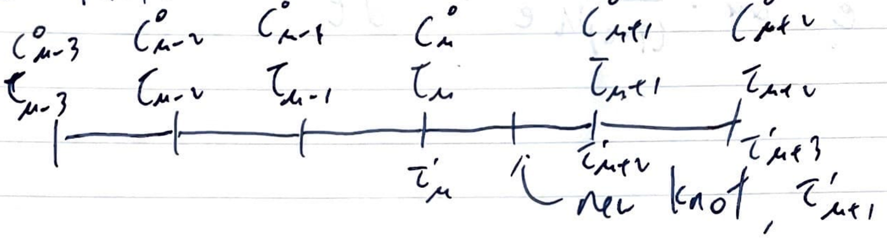
$$\therefore \boxed{b_j = \begin{cases} c_j & \text{if } 1 \leq j \leq \mu - d \\ \frac{z - \tau_j}{\tau_{j+d} - \tau_j} c_j + \frac{\tau_{j+d} - z}{\tau_{j+d} - \tau_j} c_{j-1} & \\ c_{j-1} & \text{if } \mu + 1 \leq j \leq n+1 \end{cases}}$$

No.

Date 17 · 8 · 2022

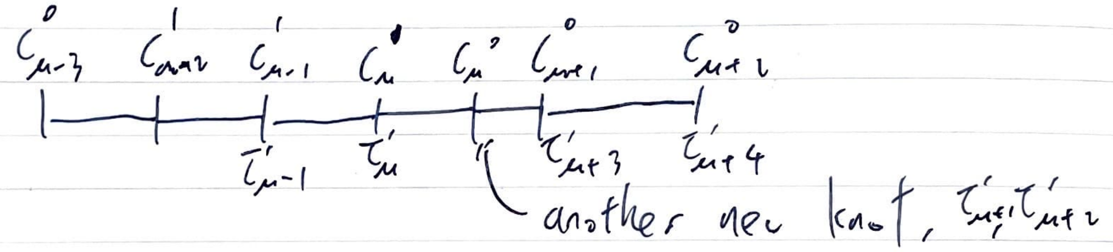
B-spline

by Böhm algorithm, we can also do knot insertion on the same knot.

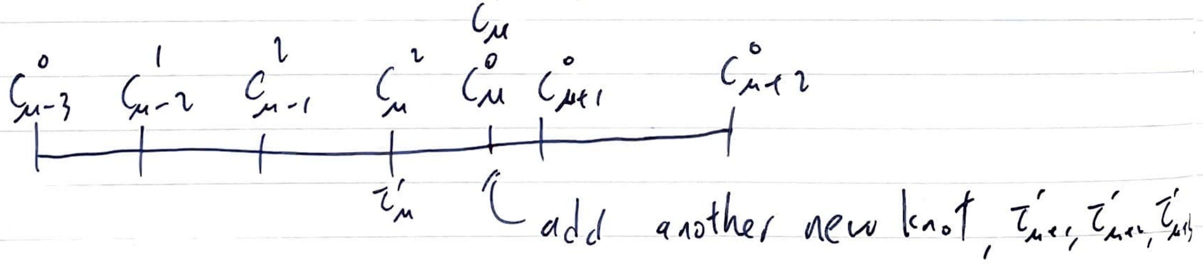


as a new knot is added, $\mu-d+1 \le i \le \mu$ are changed.

let $d=3$, so $c^0_{\mu-2}, c^0_{\mu-1}, c^0_{\mu}$ are changed.



now, $c^1_{\mu-1}, c^1_{\mu}, c^0_{\mu}$ are changed, because $\mu+1-d+1 \le i \le \mu+1$



$\therefore (\dots, c^0_{\mu-3}, c^1_{\mu-2}, c^2_{\mu-1}, c^3_{\mu}, c^2_{\mu}, c^1_{\mu}, c^0_{\mu}, c^0_{\mu+1}, c^0_{\mu+2}, \dots)$

$\therefore \quad c^0_{\mu-3} \quad c^0_{\mu-2} \quad c^0_{\mu-1} \quad c^0_{\mu} \quad c^0_{\mu+1} \quad c^0_{\mu+2}$

$c^1_{\mu-2} \quad c^1_{\mu-1} \quad c^1_{\mu}$

$c^2_{\mu-1} \quad c^2_{\mu}$

c^3_{μ}

No.

Date 17 · 8 · 2022

B-spline

$b = Ac$, by Böhm algorithm.

$A \in M_{n+1,n}$

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \mu+1 \\ \\ n+1 \end{array} \begin{pmatrix} 1 & 0 & & & & & & & \\ 0 & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & 0 & & & & \\ & & & 1-\lambda_{\mu-d+1} & \lambda_{\mu-d+1} & & & & \\ & & & & & \ddots & & & \\ & & & & & \lambda_{\mu-1} & 0 & & \\ & & & & & 1-\lambda_{\mu} & \lambda_{\mu} & & \\ & & & & & 0 & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \\ & & & & & & & 0 & 1 \end{pmatrix}$$

(μ and n written above the columns)

$$\lambda_i = \frac{z - \tau_i}{\tau_{i+d} - \tau_i}, \quad \mu-d+1 \le i \le \mu$$

$\therefore$ by repeating knot insertion,

$A = A_{m-n} A_{m-n-1} \cdots A_1$ for $\tau = (\tau_i)_{i=1}^{n+d+1}$ to $t = (t_i)_{i=1}^{m+d+1}$

$A_1 \in M_{n+1,n}$

$A_2 \in M_{n+2,n+1}$

$\vdots$

$A_{m-n} \in M_{m,m-1}$

note that for A_k, $\sum_i \alpha_i(j) = 1, \forall j \ \forall k$

sum of row elements in $A_{k+1}A_k$ is also one,

because $(A_{k+1}A_k)_{ij} = a^{k+1}_{i,\ell} a^{k}_{\ell,j}$

$= a^{k+1}_{i,i} a^{k}_{i,j} + a^{k+1}_{i,i+1} a^{k}_{i+1,j}$

$$\sum_j (A_{k+1}A_k)_{ij} = \sum_j a^{k+1}_{i,i} a^{k}_{i,j} + \sum_j a^{k+1}_{i,i+1} a^{k}_{i+1,j} = a^{k+1}_{i,i} + a^{k}_{i,i+1} = 1$$

No.

Date 17 · 8 · 2022 B-spline

by induction, sum of row in $A = 1$ //